Exercises for Chapter 5, Universality

The main message of Chapter 5 is that systems near the critical point have higher-level universal features that do not depend on some lower-level details. Recall that besides optimality (Chapter 4), universality (Chapter 5) is the second main consequence of criticality.

To illustrate universality, the book describes how the Ising model of magnetism has the same critical behavior whether it is implemented on a square lattice or on a triangular lattice (pages 95-97). In these exercises, we will investigate this claim of universality through simulations of the Ising model on both types of lattice. We will see if the models can show similar macroscopic behavior at the critical point despite differences in the microscopic details of their lattices.
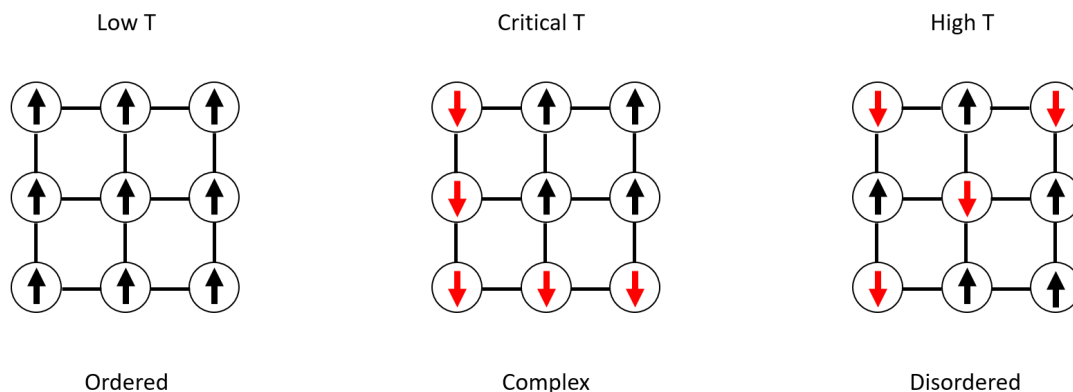
These exercises will be organized in stages. First, we will use simulations of the square-lattice Ising model to take an intuitive look at how it behaves under three different temperatures (low, critical, high). Second, we will use these movies to crudely estimate the critical temperature. Third, we will look at spatial correlations in the square-lattice model, seeing if long-range correlations emerge at the critical point. Fourth, we will plot the energy and the specific heat as a function of temperature. These functions will allow us to see the phase transition and more accurately estimate the critical temperature of the model. Fifth, we will repeat these exercises with the triangular-lattice model. We expect to find that the decay of covariance with distance at the critical point will be the same despite the lattice differences. This will lead to similar critical exponents for both models. Sixth, we will explain the energy equation and the Metropolis algorithm of the Ising model for those who want slightly more details.

1. An intuitive look at the Ising model (square-lattice)

The Ising model of magnetism in a piece of iron is explained in the book on pages 95-97. Here, I will briefly go over some of its features so the reader can get intuition about how it behaves.

The setup: The Ising model consists of a square lattice with a spin at every lattice site (see figure below). Each spin is like a tiny bar magnet that can point either up or down. For the square lattice, a spin at a given lattice site is connected to four of its nearest neighbors located in positions North, South, East and West of it. The connections between lattice sites cause the spins to tend to align with each other; if the central spin is pointing up, it will favor each of its nearest neighbors pointing up also. Likewise, if it is pointing down it will influence its neighbors to point down. We can add heat to the model by raising the temperature; this causes it to change phases as we will now explain.

The three regimes: When the temperature is low, the nearest neighbor interactions dominate, and all the spins point in the same direction. Since the spins are like tiny bar magnets all pointing in the same direction, they add together, and the sample has a net magnetization. If this is a cool piece of iron, it will stick nicely to your steel refrigerator door. This is the ordered phase and can be considered **subcritical**.

Low T	Critical T	High T

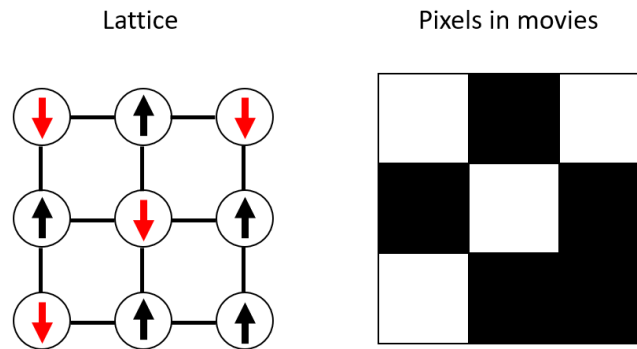| Ordered	Complex	Disordered

The three regimes of the Ising model. The model consists of a square lattice where each site hosts a spin that can be pointed either up or down. The connections between lattice sites energetically favor nearest-neighbor spins that point in the same direction, causing order. Thermal energy causes disorder, and increases with temperature, T. **Left**, at low temperature, nearest neighbor interactions dominate and all spins point in the same direction. Far **right**, at high temperature, thermal energy jostles the spins and overcomes the nearest-neighbor interactions, causing disorder. **Middle**, at the critical temperature the nearest neighbor interactions are exactly opposed by the thermal energy and there is a perfect balance of order and variety. This causes local pockets of order, but global variation, leading to complex structures.

When the temperature is raised a bit, this adds thermal energy to the system. You can think of this as jostling the spins around so that they are not all aligned anymore. Some of them will point in directions opposite to those of their nearest neighbors. The thermal energy disrupts the order we saw previously. If we are at the critical temperature, then there is a perfect balance between the order caused by the nearest neighbor interactions and the disorder caused by the thermal energy. There will be a mixture of local order and global variety, leading to complex structures. Our small sample of iron now has considerably less net magnetization and it will no longer stick to the refrigerator door. This is the phase transition region and can be considered **critical**.

When the temperature is raised yet more, thermal energy dominates. While the nearest neighbor interactions are still there, they are overwhelmed by the thermal jostling and it is very difficult to find a sizeable domain of several spins pointing in the same direction. Rather, it seems that each spin is just randomly pointing either up or down. Moreover, the spins are constantly flipping in time. This is the disordered phase and can be considered **supercritical**.

2. Watching movies and finding the critical temperature for the square lattice

To get a better idea of each of these regimes, it helps to watch movies of how the Ising model moves toward equilibrium. In these movies, a spin pointed up will be represented by a black pixel; one pointed down will be white.

Lattice   Pixels in movies

In the movies of the Ising model as it settles into equilibrium, spins pointing up are represented by black pixels, and spins pointing down are represented by white pixels.

For each movie, we will simulate the lattice from an initially disordered starting position. Next, a single lattice site will be chosen at random. The spin will then be flipped. If it lowers the energy of the overall system, then the spin flip will be retained (it can lower the energy by pointing in the same direction as its nearest neighbors, if it was pointing in the opposite direction previously). If it raises the energy of the system, the spin flip will be discarded, unless the temperature is high (we will explain this in more detail, with math, in the next section). In general, this means that when the model is at low temperatures it will converge toward an ordered state and when it is at higher temperatures it will remain a disordered state.

<mark>A. Exercise</mark>: To view a movie of the model converging to an ordered state, use the `IsingSquare` function as follows:

```
T = 0.05;
r = 20;
timesteps = 200000;
IsingSquare(T, r, timesteps);
```

This will simulate an Ising model on a 20 by 20 square lattice at a low temperature of 0.05 for 200,000 time steps. It should take only about 1-2 minutes and then the movie will play. You will see the movie start out with a random configuration of spins, about half pointing up and half pointing down. This will rather quickly converge to a state where all the pixels are either white (pointing down) or black (pointing up). This is a representative low temperature, ordered phase. Note that sometimes at low temperatures the system can settle into a "stripe" where half the sites are pointed up and half the sites are pointed down. While this is lower energy than the random
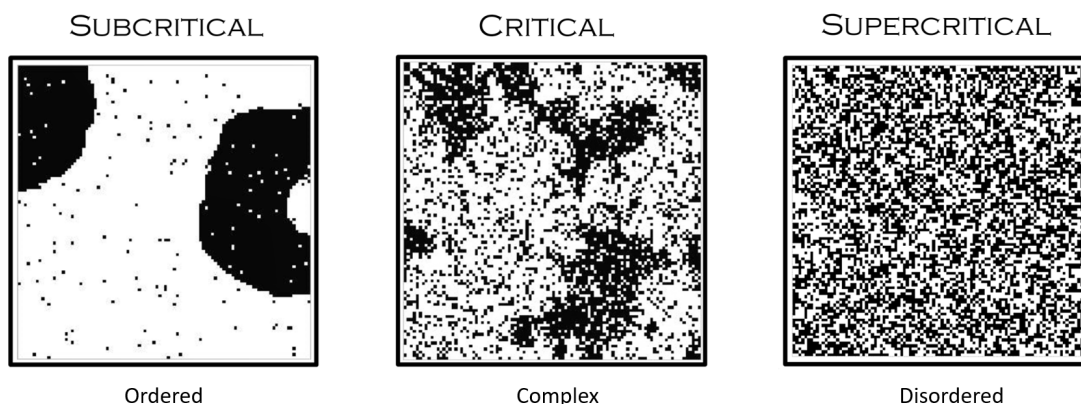
configuration, it is not the lowest energy. In a small simulation like this, such a local minimum (as opposed to a global minimum) occasionally happen.

Next, run the function again, but this time try a much higher temperature:

```
T = 8.00;
r = 20;
timesteps = 200000;
IsingSquare(T, r, timesteps);
```

Here you should see it remain flickering in a disordered state.

Finally, try to find the critical temperature (T to the nearest quarter of an integer; search between 0 to 8) by looking for a movie that is neither highly ordered nor completely disordered. Near T_{crit}, you should see blobs of many sizes on the screen, slowly moving about, extending tendrils and breaking them off to form new blobs. This should appear somewhat life-like near the critical temperature, with blobs appearing like amoeba under a microscope. The figures below can give you a hint of what to look for.

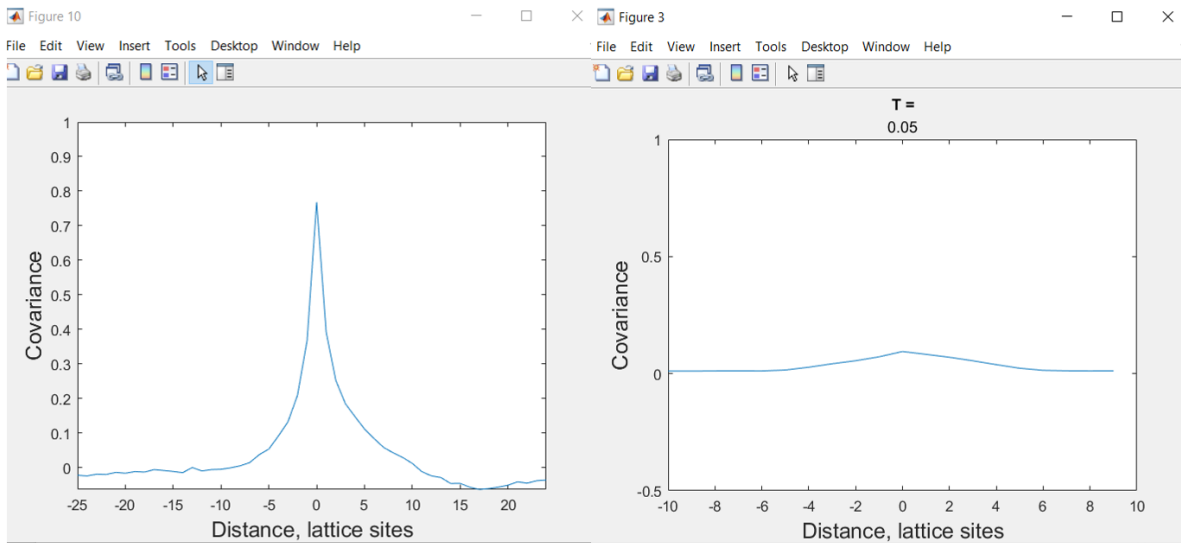| SUBCRITICAL | CRITICAL | SUPERCRITICAL |
| Ordered | Complex | Disordered |

Representative frames from movies of the Ising model. **Left**, when the model is subcritical, nearest neighbor connections promote order and the system converges to a state where all spins are pointed either up or down. The movie frame shown indicates it is moving toward all spins pointed down in this run. **Middle**, when the model is nearly critical, groups of spins pointing in the same direction occur at various sizes. For example, note the presence of many small, several medium, and a few large domains of spins pointed up. In the movie, these blob-like structures persist and change shape, seeming to extend tendrils and retract them, almost like amoeba. Activity is complex here. **Right**, when the model is supercritical, thermal energy dominates the nearest neighbor connections and causes disorder. There are always about half the spins pointing up and half down; each spin flips many times in a second so no large structures persist.

For the square Ising model on a 20 by 20 lattice, what does the critical temperature seem to be, based on your assessment of a life-like movie?

3. The covariance length at different temperatures for the square lattice

You may have also noticed that the program `IsingSquare` produces a second plot in addition to the movie. This second plot shows the covariance between lattice sites as a function of distance. Two such covariance plots are shown in the figure below.

Representative plots of covariance as a function of distance for two runs of IsingSquare at two different temperatures. Note that covariance is higher for short distances and then gradually falls off as you go further away. The Ising model should show the largest covariance with the slowest decline when it is near the critical temperature.

B. Exercise: Compare the covariance plots at the low, high and nearly critical temperatures you used earlier. Which has the slowest decline in covariance with distance?

Interestingly, this extension of the covariance is a simple signature of emergence. In the model, we have only built in nearest neighbor connections. If that were all that mattered, then we should not see covariance between lattice sites that are not directly connected to each other. Yet we do see this near the critical temperature. This means that events or structures emerge in the lattice that have a scale larger than the length of the nearest neighbor connections. Lattice sites that are very far away from each other can still be statistically related when the system is tuned to be near the critical point.

So far, we are only looking for approximate agreement between our plots and the critical temperature. Next, we will try to be more precise.

4. Determining the critical temperature, T_{crit}, for the square lattice

To more accurately determine T_{crit}, we will now systematically sweep through the temperatures and identify where phase transition occurs.

Run the short script IsingLoopSquare shown below:

```
%This section generates several files.
tic;
timesteps = 50000; %you can increase the timesteps to get more accurate results, but
it will take longer
r = 10;           %you can increase the dimensions of the lattice for more accuracy,
but this will also take longer
numRuns = 20;
increment = 0.25;  %you may want to use a smaller increment when you are probing the
critical temperature in more detail
for i=1:numRuns
    i
    TeeVals(i) = (increment * i);
    IsingSquare(TeeVals(i), r, timesteps);
end
toc
```

Note that it sweeps the temperature from 0.25 to 5.00 in steps of 0.25. Thus, twenty different temperatures will be tried, resulting in a movie and a covariance plot for each. You can examine all the covariance plots to try to see which ones fall off most gradually.

But how can we find the T_{crit} more accurately from this sweep? Recall that near the phase transition point, there should be functions of the system that show sharp peaks. This is illustrated frequently in Chapter 4, where the information transmission (Figures 4.4, 4.9), dynamic range (Figures 4.6, 4.8) and susceptibility (Figures 4.5, 4.10) show peaks near the critical point. With the Ising model, there are analogous functions we can look at. For example, the susceptibility of the Ising model is related to how many spins will flip in response to flipping a single spin. *The susceptibility peaks near the critical point*
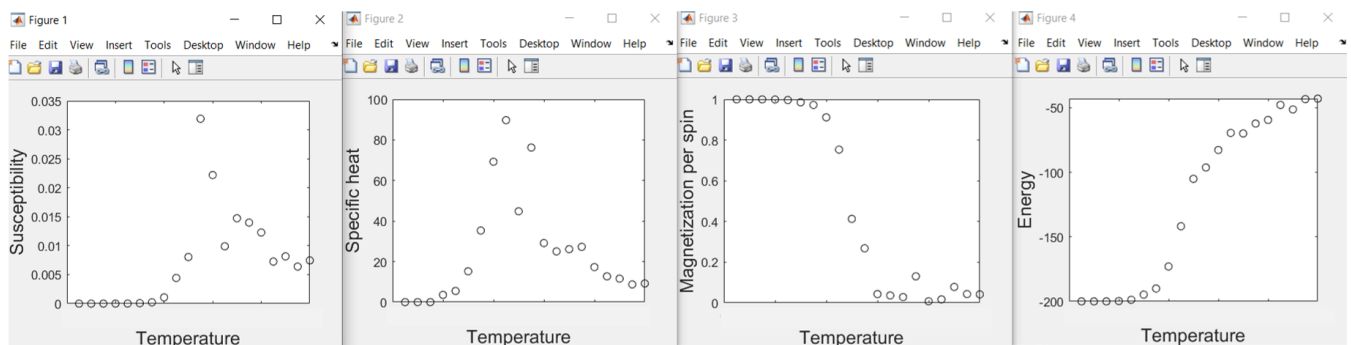
In addition, there is something called the specific heat. This is the change in the energy for a given change in temperature. Like the susceptibility, this is largest right near the critical point. *Specific heat peaks near the critical point*.

Another signature of the phase transition is that the order parameter changes at the boundary between phases. For the branching model, the order parameter is the density of active sites, or just the fraction of nodes that are on, which can range from 0 to 1 (Figures 3.3, 4.5). *The density of active sites shows a transition near the critical point.*

For the Ising model, the order parameter is closely related to this, and is just the magnetization, M. For our purposes, the magnetization is just the sum of all the spins, divided by the total number of spins. Recall that each spin can be either up (+1) or down (-1). If all the spins are pointed up, then they will sum together to produce a positive number close to 1 (when normalized by dividing by the total number of spins). At low temperatures we should expect M to be high, while at high temperatures it should drop to nearly zero. Note that it will be equally likely for all the spins to point down; for clarity we avoid looking at this by just taking the absolute value. *Magnetization shows a transition near the critical point.*

Finally, we can look at the energy of the Ising model. This will be lowest when all the spins are pointing in the same direction because the nearest neighbor connections are in harmony with the arrangement of spins. The energy will be highest when the spins are pointing in different directions and about half of the nearest neighbor connections will not be in harmony. *Energy shows a transition near the critical point.*

We therefore have several functions we can use to more carefully identify the critical point: (1) the susceptibility (peaks), (2) the specific heat (peaks), (3) the magnetization (transitions), and (4) the energy (transitions). To make plots of these variables as a function of temperature, run the script `IsingLoopSquare` and wait until it finishes. After that, run the script `IsingPlotterSquare` which will produce four plots of the variables. Without giving away the critical temperature, you should see something roughly like these:
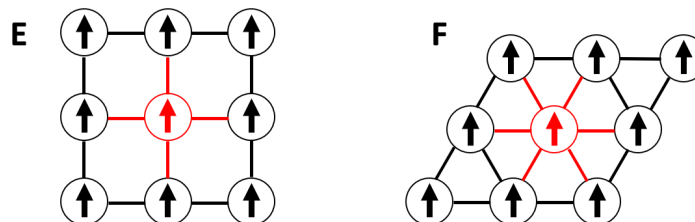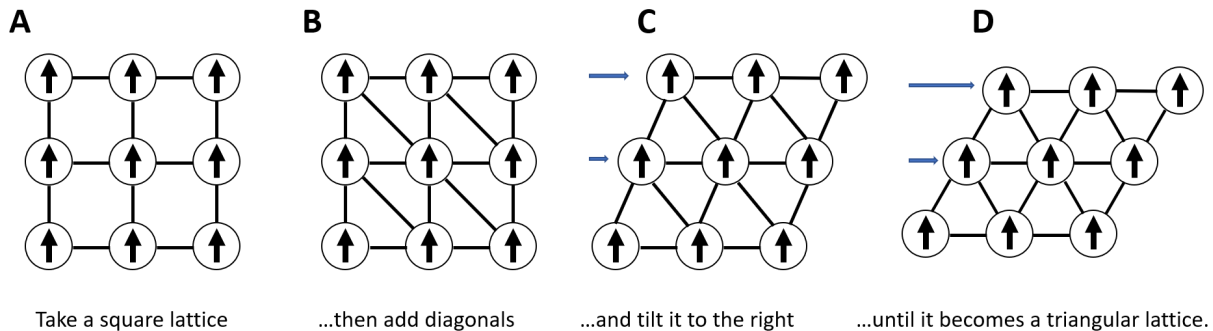

Representative curves produced after running `IsingLoopSquare` and `IsingPlotterSquare` for a small 10 by 10 square lattice for only 50,000 time steps. Note that the susceptibility and the specific heat show peaks, while the magnetization and the energy show transitions. The critical temperature can be identified approximately by these peaks and by where the transitions occur. A more precise value of T_{crit} can be found by running larger lattices for more time steps.

Running both programs will probably take 5 minutes or more on a moderately good laptop (mine has an Intel Core i7-8750H CPU @ 2.20GHz processor with about 16 GB of RAM). To get a more accurate estimate of the critical temperature for this model, you can run larger lattices (100 by 100) for longer times (e.g., 2,000,000 timesteps). This will of course take more time and can be done with finer precision by making the variable `increment` in `IsingLoopSquare` as small as 0.1 and running only near the region where you suspect the transition occurs.

C. Exercise: With this information in hand, what is your best estimate of T_{crit} for the Ising model on a square lattice? Show plots to justify your answer.

5. The Ising model on the triangular lattice: T_{crit} and the covariance length

As mentioned in the book, the Ising model can also be run on a triangular lattice. How can we think of this in relation to the square lattice? The figure below shows how we can transform the square lattice into a triangular lattice by adding a few connections and tilting the lattice.

A	B	C	D
Take a square lattice	…then add diagonals	…and tilt it to the right	…until it becomes a triangular lattice.

A site in the square lattice has four nearest neighbors, while in the triangular lattice it has six.

Transforming the square lattice to the triangular lattice of the Ising model. **A**, The square lattice, where each site has nearest neighbor connections to four other sites, located North, South, East and West from the original site. Notice that the lattice seems to be made up of a single square, with spins at each corner, that could be tiled over the entire plane. **B**, To transform into a triangular lattice, the first step is to add diagonal connections from upper left to lower right. **C**, To see that this is topologically equivalent to a triangular lattice, we now begin to tilt the lattice to the right. **D**, If we tilt it the proper amount, we can form equilateral triangles with the connections. Notice that this lattice seems to be made up of a single triangle, with spins at each corner, that could be tiled over the entire plane. **E**, In the square lattice, each site has four nearest neighbors, with connections to them shown in red. **F**, In the triangular lattice, each site has six nearest neighbors, with connections to them shown in red. Although the lattices have different microscopic structures, will they show similar emergent behavior when near the critical point?

In the triangular lattice, each site has six nearest neighbors now instead of four. We might expect that this would change T_{crit}. The reasoning goes as follows: the critical temperature is just the point where the thermal energy (promoting disorder) is exactly balanced by the nearest neighbor energy (promoting order). In the square lattice, the thermal energy is fighting against four connections per spin, while in the triangular lattice it is fighting against six connections per spin.

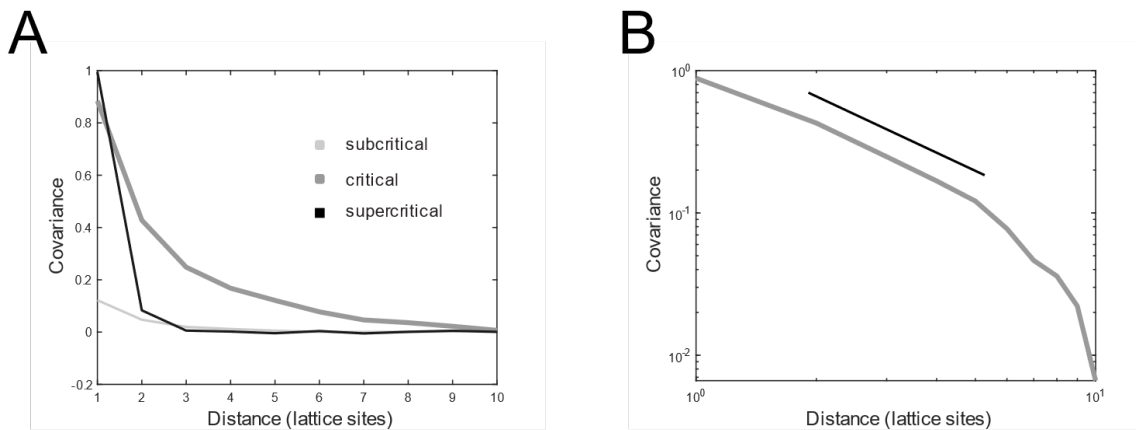D. Exercise: Based on this, what do you predict the new T_{crit} will be?

To see if your guess is close to correct, run the script `IsingLoopTriangle` followed by `IsingPlotterTriangle`. The numbers currently loaded will set up a simulation with a

10 by 10 lattice, running for 50,000 time steps. As before, this rough and quick calculation can give you an approximate idea of where T_{crit} lies. For more accuracy, run larger lattices for longer times, just as you did with the square lattice.

As you might have guessed by now, the square lattice and the triangular lattice do not have the same critical temperatures. So, this means that the microscopic details can affect the behavior, right? Yes, they can. But universality has a claim that is slightly different than this – it argues that the emergent behaviors of both lattices should be the same when they are both near the critical point. This means for example that the decay of the covariance functions should look similar when each lattice is at its respective critical temperature.

E. Exercise: To test this claim, run larger versions of both the square lattice and the triangular lattice models for longer times and compare their covariance plots. If you make the axes logarithmic for both scales, you should see something approaching a power law in both plots (see the figure below; at the end of this section there is a series of figures showing you how to change to logarithmic axes in Matlab). Are the slopes the same within some margin of error? Show plots for both lattices in logarithmic scales and estimate their slopes (exponents). A sample of something like what you should get is shown below.

Covariance length is maximized at the critical point. **A,** Covariance plotted against distance between spins for three Ising model regimes. Covariance has the longest range when the model is near the critical temperature. **B,** Covariance from the critical temperature plotted in log log coordinates, showing a long-tailed distribution that would approach a power law for larger simulation sizes (shown for a square lattice of 100 x 100 spins.)

Note that the variable `CenteredAR` contains the covariance values and is saved with each file after an Ising model is run with a particular set of parameters. To make covariance plots, you can paste the code below into the Matlab command line:

```
if mod(r, 2) == 0
    figure; plot([-(floor(r/2)):1:(floor(r/2)-1)], CenteredAR); ylabel('Covariance', 'FontSize', 15);
    xlabel('Distance, lattice sites', 'FontSize', 15); ylim([-0.5 1]); title('T =', T);
else
    figure; plot([-(floor(r/2)+1):1:(floor(r/2)-1)], CenteredAR); ylabel('Covariance', 'FontSize', 15);
    xlabel('Distance, lattice sites', 'FontSize', 15); ylim([-0.5 1]); title('T =', T);
end
```
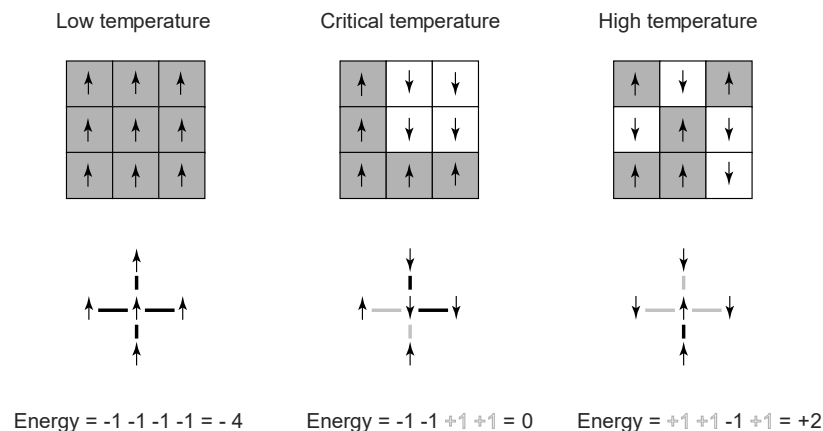
You can then transform the axes from linear to logarithmic scales by using the steps given at the end of this document. Doing this should help you to create plots like those shown in figures A and B above.

6. The square lattice Ising model explained in slightly more detail

This section is somewhat redundant but will introduce the energy equation and the Metropolis algorithm that underlie the model.

The energy equation. As we stated earlier, the Ising model is composed of a set of spins, each acting like a tiny bar magnet, that can point either up or down. These spins populate a regular square lattice, as shown in the figure below. Each spin is connected only to its nearest neighbors in the North, South, East and West lattice sites.

Low temperature   Critical temperature   High temperature

Energy = -1 -1 -1 -1 = - 4   Energy = -1 -1 +1 +1 = 0   Energy = +1 +1 -1 +1 = +2

The Ising model energy. **Upper row** shows regular 3 × 3 square lattices with spins (arrows) occupying each square. The spins may point either up or down; energy is *subtracted* for same orientation spins and *added* for different orientation spins. Temperature increases from left to right, producing increasing levels of disorder. **Middle row** highlights the spin from the center of the lattice and the connections to its nearest-neighbor spins. Black connections indicate spin pairs with similar directions, gray connections indicate spins pairs with opposing directions. **Lower row** shows energy calculations just for the central spin and its nearest neighbors. Each black (same orientation) connection contributes -1 unit, each gray (different orientation) connection contributes +1 unit. Highly ordered states have low energy, disordered states have high energy.

When two connected spins point in the same direction, they have negative energy. Since systems tend to flow toward low energy states, this configuration will be favored. In contrast, when two connected spins point in opposite directions, they have positive

energy; this will not be favored by the system. From this, we can see that the nearest neighbor connections will tend to make the spins all align in the same direction, as this will be the lowest energy state. For the entire population of spins, we can calculate the energy, E, from this simple equation which is just the sum of the energies of all the connected spin pairs:

$$E = \sum_{i \neq j} -J_{i,j} \sigma_i \sigma_j$$

Here, $-J_{ij}$ is the connection strength between spin i and spin j. Note that $-J_{ij}$ will always be negative one in the Ising model. Here, σ_i and σ_j are the orientations of spins i and j, respectively, and can be either (+1, up) or (-1, down). The sum is taken over all nearest neighbor pairs in the model and does not allow any spin to interact with itself (hence the restriction $i \neq j$). As we will see next, the model will evolve from an initially random configuration toward lower, more negative, energy configurations as it moves toward equilibrium.

The Metropolis algorithm. Let us first consider what will happen if we start from a random configuration where about half the spins are pointed up and half are pointed down. To simulate the evolution of the model toward the equilibrium state when the temperature is zero, we follow a simple procedure. First, randomly select a spin and provisionally flip it. Second, calculate the change in energy that would occur if that spin flip were retained. Third, if the spin flip lowers the energy, then accept it. If the spin flip raises the energy, then discard it. If this procedure is repeated many times, the model will move toward equilibrium. In this case, the energy will be maximally negative, and the spins will become completely aligned, pointing either all up or all down. This is the highly ordered state where nearest neighbor connections dominate. Notice that from an initially random configuration, it has a 50% chance of evolving toward an all up or an all down configuration.

What will happen if we now add thermal energy to the model? We can do this by raising the temperature, T, above zero. Thermal energy will jostle the spins, causing some of them to randomly flip in directions opposed to their neighbors. In this situation, there will be a competition of sorts between the nearest neighbor connections, promoting order, and the thermal energy, promoting disorder. We will indicate the amount of thermal energy by the temperature, T.
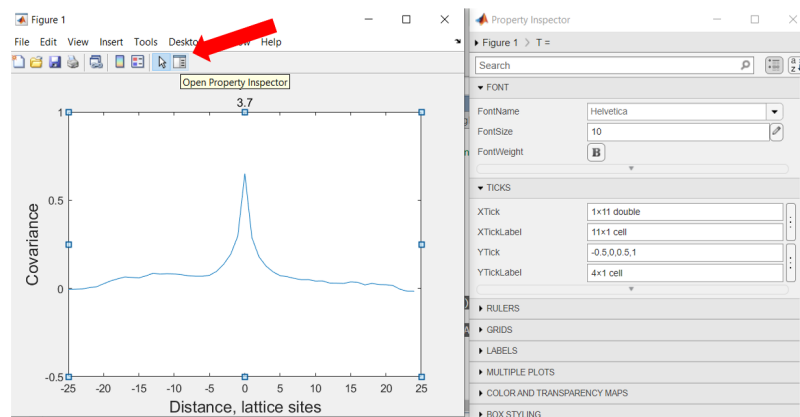
Now that we have thermal energy, our procedure for simulating the evolution of the model toward an equilibrium state will have to be slightly altered. First, as before, randomly select a spin and provisionally flip it. Second, calculate the change in energy that would occur if that spin flip were retained. Third, if the spin flip lowers the energy, then accept it. But if the spin flip raises the energy, accept it with some probability that depends on T. When T is high, this probability is high; when T is low, it is lower. Repeat this process until the energy of the model does not change. More specifically, the probability P of accepting a spin flip (that raises energy) at temperature T is given by $P = e^{-\frac{\Delta E}{kT}}$. Here ΔE is the change in energy caused by the spin flip and k is Boltzmann's

constant (which in our simulation is just set to 1 for ease of calculation). Notice that as T increases, P increases also. At high T, spin flips that raise the energy are more likely. Also, when T = 0, P = 0.
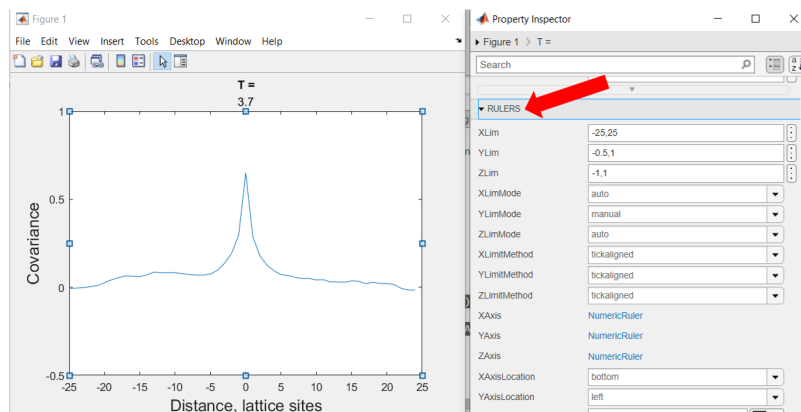
This procedure for flipping the spins in the Ising model is called the Metropolis algorithm and is implemented in the Matlab codes. More generally, it is a type of Monte Carlo approach that relies on randomness. A more detailed discussion of the Metropolis algorithm and Monte Carlo methods can be found in (*Elements of Phase Transitions and Critical Phenomena*, by Nishimori and Ortiz, 2011).
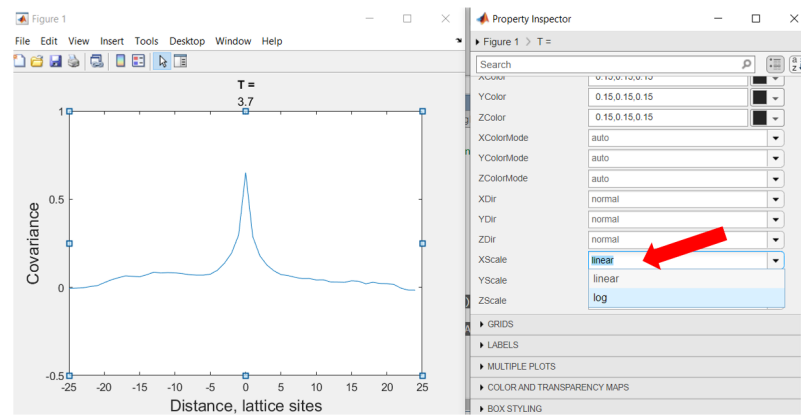
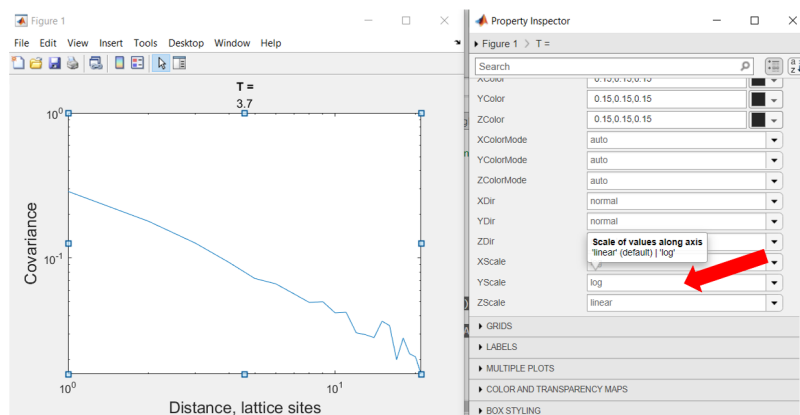How to change a Matlab figure from linear to logarithmic axes:

Once you have a covariance plot, open the Property Inspector.

Within the Property Inspector, click on RULERS.

Go down to XScale and select log; do the same for YScale.

You should now have a plot of the covariance in logarithmic axes. If the system is near the critical point, you should see an approximate power law. Note that only the positive (right hand) side of the covariance plot is shown, as you can't usually take logarithms of negative numbers.

Matlab code used for exercises in this chapter, listed in order of use:

IsingSquare
IsingLoopSquare
IsingPlotterSquare
IsingTriangle
IsingLoopTriangle
IsingPlotterTriangle